First-price auction with n bidders

Consider a first-price auction with n bidders, whose valuations are ordered as

$$v_1 > v_2 > v_3 > \cdots > v_n$$

In the event of a tie, the object is awarded to the bidder with the highest valuation.

1. Find all Nash equilibria. It is convenient to characterize them rather than express them numerically
2. Indicate whether in any of them the winner of the auction is a bidder other than the one with the highest valuation
3. Which group of equilibria do you think is the one most likely to arise in practice?

Solution

1. Let b_i denote bidder i's bid, and let $b = (b_1, \dots, b_n)$ be a bid profile

   In a first-price auction, bidder i's payoff is

$$u_i(b) = \begin{cases} v_i - b_i & \text{if bidder } i \text{ wins} \\ 0 & \text{if bidder } i \text{ loses} \end{cases}$$

   We characterize the *pure-strategy* Nash equilibria

   First, in any equilibrium, bidder 1 must win

   Indeed, suppose some bidder $k \neq 1$ wins with bid p

   If $p < v_1$, then bidder 1 can deviate to a bid slightly above p, win the object, and obtain a strictly positive payoff

   If instead $p \geq v_1$, then bidder k is paying at least v_1, and since $v_k < v_1$, bidder k gets strictly negative payoff and would rather deviate and lose, obtaining payoff 0

   Therefore, no bidder other than 1 can win in equilibrium

   Now let

$$p = \max_{j \geq 2} b_j$$

   Since bidder 1 must win, we must have

$$b_1 \geq p$$

   If $b_1 > p$, then bidder 1 could lower her bid slightly and still win, increasing her payoff

   Hence, in equilibrium it must be that

$$b_1 = p$$

   and therefore at least one other bidder must also bid p

   Next, we show that p must satisfy

$$v_2 \leq p \leq v_1$$

   If $p < v_2$, then bidder 2 could deviate to a bid slightly above p, still below v_2, win the auction, and obtain a strictly positive payoff

   So we must have

$$p \geq v_2$$

   If $p > v_1$, then bidder 1, who wins at price p, obtains negative payoff and would rather deviate and lose

   So we must have

$$p \leq v_1$$

   Conversely, any bid profile satisfying the following conditions is a Nash equilibrium:

$$b_1 = p$$

$$\max_{j \geq 2} b_j = p$$

$$v_2 \leq p \leq v_1$$

$$b_j \leq p \quad \text{for all } j \geq 2$$

with bidder 1 winning by the tie-breaking rule

To verify this:

- bidder 1 gets payoff $v_1 - p \geq 0$, so she does not want to lose, and bidding more only lowers her payoff
- any bidder $j \geq 2$ can only win by bidding more than p, but since $p \geq v_2 > v_j$ for every $j \geq 3$, and also $p \geq v_2$ for bidder 2, such a deviation would give non-positive payoff
- losing bidders already obtain payoff 0, so they do not gain from lowering their bids

Therefore, the full set of pure-strategy Nash equilibria is given by all bid profiles in which bidder 1 wins, her bid is tied by at least one other bidder, and the common highest bid p lies in the interval

$$v_2 \leq p \leq v_1$$

Equivalently, all pure Nash equilibria are the bid profiles such that

$$b_1 = p, \quad \max_{j \geq 2} b_j = p, \quad v_2 \leq p \leq v_1$$

with bidder 1 receiving the object by the tie-breaking rule

Example 1

Suppose $n = 3$ and valuations are

$$v_1 = 10, \qquad v_2 = 7, \qquad v_3 = 4$$

Consider the bid profile

$$(b_1, b_2, b_3) = (7, 7, 0)$$

Here the highest bid is $p = 7$, which satisfies

$$v_2 = 7 \leq p = 7 \leq 10 = v_1$$

Bidder 1 wins by the tie-breaking rule and obtains payoff

$$10 - 7 = 3$$

Bidder 2 loses and gets payoff 0. If bidder 2 raises her bid above 7, she would have to bid more than her own valuation and would get a negative payoff if she won

Bidder 3 also cannot profitably deviate, because any winning deviation requires bidding above 7, which is greater than $v_3 = 4$

Thus,

$$(7, 7, 0) \text{ is a Nash equilibrium}$$

Example 2
With the same valuations

$$v_1 = 10, \qquad v_2 = 7, \qquad v_3 = 4$$

consider instead

$$(b_1, b_2, b_3) = (9, 9, 1)$$

Again, bidder 1 wins by the tie-breaking rule, and the common highest bid is $p = 9$, which satisfies

$$v_2 = 7 \leq p = 9 \leq 10 = v_1$$

Bidder 1's payoff is

$$10 - 9 = 1$$

Bidder 2 loses and gets payoff 0. Although bidder 2 is bidding above her own valuation, she still has no profitable deviation from this profile: lowering the bid keeps her losing with payoff 0, while raising it to win would make her payoff negative

The same is true for bidder 3

Therefore,

$$(9, 9, 1) \text{ is also a Nash equilibrium}$$

These two examples illustrate that the equilibrium set includes both the undominated equilibrium with $p = v_2$ and other equilibria with $p > v_2$

2. No

   From part 1, in every pure-strategy Nash equilibrium bidder 1 must win the auction

   To see this again briefly, suppose instead that some bidder $k \neq 1$ wins with bid p

   If

   $$p < v_1$$

   then bidder 1 can profitably deviate by bidding slightly above p, winning the object and obtaining a strictly positive payoff

   If

   $$p \geq v_1$$

   then bidder k pays at least v_1, and since

   $$v_k < v_1$$

   bidder k's payoff is negative, so bidder k would rather deviate and lose, obtaining payoff 0

   Thus, no equilibrium can have a winner different from bidder 1

   Therefore, in every Nash equilibrium the object is won by the bidder with the highest valuation

3. Among the Nash equilibria characterized in part (a), the ones most likely to arise in practice are those in which the winning bid is

$$p = v_2$$

that is, the equilibria in which bidder 1 wins while the highest losing bid is exactly the second-highest valuation

The intuition is the following

If the common highest bid satisfies

$$p > v_2$$

then bidder 2 is bidding above her own valuation whenever she is one of the bidders tying at p, which is weakly dominated, since winning at such a bid would give her a negative payoff

More generally, equilibria with

$$p > v_2$$

require some bidders to submit unnecessarily high bids that are not credible in practice, because they expose them to losses if the tie-breaking rule or the opponents' behavior changes slightly

By contrast, when

$$p = v_2$$

the outcome is more robust

- bidder 1 wins and obtains payoff $v_1 - v_2$
- bidder 2 can bid v_2 and lose with payoff 0
- no bidder is bidding above their own valuation

Thus, the most plausible equilibria are the *undominated* ones, namely those in which bidder 1 wins with bid v_2, and at least one other bidder, typically bidder 2, also bids v_2

Therefore, the equilibria most likely to arise in practice are the undominated equilibria with highest bid $p = v_2$

An equilibrium is called *undominated* when no player is using a weakly dominated strategy in that equilibrium

In the context of this auction, this matters because some Nash equilibria exist only because certain bidders submit bids that are technically best responses at that exact bid profile, but are still poor strategies more generally